New QES Hermitian as well as non-Hermitian PT invariant Potentials

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We start with quasi-exactly solvable (QES) Hermitian (and hence real) as well as complex PT-invariant, double sinh-Gordon potential and show that even after adding perturbation terms, the resulting potentials, in both cases, are still QES potentials. Further, by using anti-isospectral transformations, we obtain Hermitian as well as PT-invariant complex QES periodic potentials. We study in detail the various properties of the corresponding Bender-Dunne polynomials.

I. INTRODUCTION

In recent years, various features of the complex PT-symmetric Hamiltonians have been explored in the literature,see for example Refs. [1–4]. In particular, it has been shown that so long as the PT-symmetry is not spontaneously broken, then the energy eigenvalues of the Schrödinger equation are real. Further, several PT-symmetric complex QES[5, 6] potentials have been discovered [8–10]. The purpose of this paper is to point out that even after suitably perturbing either a Hermitian or a complex PT-invariant QES potential, one can still obtain another Hermitian or complex, PT-invariant, QES potential.

II. PT SYMMETRIC NON-HERMITIAN QES SYSTEM

A. The Model

We start from the well known non-Hermitian but PT-invariant potential $-[a\cosh(2x)-iM)^2$, $(\hbar = 1 = 2m)$ [8] and show that even the perturbed complex PT-invariant Hamiltonian

$$H = p^2 - [a\cosh(2x) - iM]^2 + \frac{l(l+1)}{\sinh^2(x)} - \frac{l(l+1)}{\cosh^2(x)}, \tag{2.1}$$

where a, l, M are real and $-1 < l < 0$ is a QES potential so long as $M-2l-1$ or $M+2l+1$ is a positive even integer. Note, the restriction $-1 < l < 0$ has been imposed so that the potential is not too singular at $x = 0$ and there is a communication from the left side to the right side.

Let us first show that the above non-Hermitian Hamiltonian (2.1) is a PT-symmetric one. We note that in this case the parity transformation is defined as $x \longrightarrow i\frac{\pi}{2}-x$, which is reflection of coordinate about the point $x = i\frac{\pi}{4}$. Under the time reversal transformation $t \longrightarrow -t$ and further, one replaces $i \longrightarrow -i$. One can check easily that, under PT $\cosh(2x) \longrightarrow -\cosh(2x)$,$\cosh^2(x) \longrightarrow -\sinh^2(x)$ and $\sinh^2(x) \longrightarrow -\cosh^2(x)$.

We substitute

$$\psi(x) = e^{i\frac{a}{2}\cosh(2x)}\phi(x) \tag{2.2}$$

in the Schroödinger equation $H\psi = E\psi$ with the H given by Eq. (2.1) and obtain

$$\begin{aligned}&\phi''(x) + 2ia\sinh(2x)\phi'(x) + [(E - M^2 + a^2)\\&-2i(M-1)a\cosh(2x) - \frac{l(l+1)}{\sinh^2(x)} + \frac{l(l+1)}{\cosh^2(x)}]\phi(x) = 0\end{aligned}$$

On further substituting

$$\phi = [\cosh(x)]^{\alpha}[\sinh(x)]^{\beta}\eta(x)\,, \tag{2.3}$$

we find that the system admits non-singular QES solutions provided

$$\alpha(\alpha+1) = l(l+1)\text{and} \;\; \beta(\beta+1) = l(l+1) \tag{2.4}$$

is satisfied. The condition in Eq. (2.4) implies either of the following four conditions i.e.

(i) $\alpha = \beta = l+1$ (ii) $\alpha = \beta = -l$ (iii) $\alpha = -l, \beta = l+1$ (iv) $\alpha = l+1, \beta = -l$.

We discuss all the cases separately,

Case (i): $\alpha = \beta = l+1$,

In this case, $\eta(x)$ as given by Eq. (2.3) can be shown to satisfy

$$\begin{aligned}&\eta''(x) + 2[(l+1)\coth(x) + (l+1)\tanh(x) + ia\sinh(2x)]\eta'(x)\\&+\left[\epsilon - iza\cosh^2(x)\right]\eta(x) = 0\,,\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}\epsilon &= E - M^2 + a^2 + 4(l+1)^2 - 2ia(2l - M + 3)\\ z &= 4(M - 2l - 3)\,.\end{aligned} \tag{2.6}$$

This system has p QES solutions in case $M - 2l - 1 = 2p$ with $p = 1, 2, 3, \cdots$. These solutions are of the form

$$\eta(x) = \sum_{n=0}^{\infty} a_n (i)^n [\cosh 2(x)]^n \,, \tag{2.7}$$

Two of the low lying solutions are

$$M = 2l + 3 \,, \quad \eta = constant \,, \quad E = -a^2 + 4l + 5 \,, \tag{2.8}$$

$$\begin{aligned} &M = 2l + 5 \,, \quad \eta = A\cosh(2x) + iB \,, \quad \frac{A}{B} = \frac{E + a^2 - 12l - 21}{4a} \,, \\ &E = 8l + 15 - a^2 \pm \sqrt{(2l+3)^2 - 4a^2} \,. \end{aligned} \tag{2.9}$$

Note that since $\alpha = \beta$ the PT symmetry is unbroken and hence, as expected, the energy eigenvalues are real.

Case (ii): $\alpha = \beta = -l$.

The solutions in this case can be obtained from the solutions of case (i) by everywhere changing $l \longrightarrow -l - 1$. Thus in this case, one has p QES solutions in case $M + 2l + 1 = 2p$ with $p = 1, 2, 3, \cdots$. Note that since $\alpha = \beta$, even in this case, the PT symmetry is unbroken and the energy eigenvalues are real.

Case (iii): $\alpha = -l \,, \; \beta = l + 1$.

In this case we have p QES solutions in case $M = 2p$, $p = 1, 2, \cdots$. Two of the low lying QES solutions are

$$M = 2 \,, \quad E = 3 - a^2 - 2ai(1 + 2l) \,, \tag{2.10}$$

$$M = 4 \,, \quad E = 11 - a^2 - 2ai(1 + 2l) \pm \sqrt{(1 - a^2) - ia(1 + 2l)} \,. \tag{2.11}$$

Note that since in this case $\alpha \neq \beta$, hence the PT symmetry is broken spontaneously and eigenvalues are no more real. In the special case of $\alpha = \beta = \frac{1}{2}$, the PT symmetry is restored and E becomes real.

Case (iv): In this case, the solutions can simply be obtained from the solutions of the case (iii) by changing $l \longrightarrow -l - 1$.

Ordinarily, the boundary conditions that give the quantized energy levels are $\psi(x) \to 0$ as $|x| \to \infty$ on the real axis. However, in the present case, we have to continue the eigenvalue problem into the complex $-x$ plane[6]. On putting, $x = u + iv$ where u and

v are real, it is easy to see that for $u > 0$ the boundary condition is satisfied so long as $-\pi < v < -\frac{\pi}{2}$ (mod π) while for $u < 0$ it is satisfied if $-\frac{\pi}{2} < v < 0$ (mod π)

It is worth mentioning that, the non-Hermitian PT symmetric Hamiltonian in Eq. (2.1) after a suitable change of variable can be expressed in terms of the $SL(2,R)$ generators (at most quadratic). To show this we substitute $t = \cosh(2x)$ in the Eq. (2.5) to obtain $H_g\eta = E\eta$ where,

$$H_g = -4(t^2-1)\frac{d^2}{dt^2} - [(8l+12)t \;+\; 4ai(t^2-1)]\frac{d}{dt} - [-M^2 + a^2 \\ +\; 4(l+1)^2 - 2ai(2l - M + 3) - \frac{iz}{2}(t+1)] \quad (2.12)$$

This gauged Hamiltonian then can be expressed in terms of the generators of the $SL(2,R)$ by

$$H_g = -4[(J_0^2 - J_-^2) + ia(J_+ - J_-) \;+\; (n + 2l + 2)J_0] - [-M^2 + a^2 \\ +\; 4(l+1)^2 + 4n(l+1) + n^2] \quad (2.13)$$

while the generators are given by $J_- = \frac{d}{dt}$, $J_0 = t\frac{d}{dt} - \frac{n}{2}$ and $J_+ = t^2\frac{d}{dt} - nt$. Gauged Hamiltonian in terms of $SL(2,R)$ for a more general system has been discussed in Ref. [7].

B. Bender-Dunne (BD) Polynomials

Case (i): $\alpha = l + 1 = \beta$. We make a change of variable in Eq.(2.5), $\cosh^2(x) = t$, yielding

$$t(t-1)\eta''(t) - [(l+3/2) - (2l+3-2ia)t - 2iat^2]\eta'(t) + \\ +(1/4)[\epsilon - iazt]\eta(t) = 0 \quad (2.14)$$

On further substituting

$$\eta(t) = \sum_{n=0}^{\infty} \frac{P_n(\epsilon)t^n}{n!\Gamma(n+l+3/2)}\,, \quad (2.15)$$

yields the three-term recursion relation satisfied by the polynomials, $P_n(\epsilon)$

$$P_{n+1}(\epsilon) \;-\; [\frac{\epsilon}{4} + n(n+2l+2-2ia)]P_n(\epsilon) \\ +\; ia(n+l+1/2)n[M - 2l - 2n - 1]P_{n-1}(\epsilon) = 0\,. \quad (2.16)$$

First few Polynomials are:

$$\begin{aligned} P_0(\epsilon) &= 1\,, \\ P_1(\epsilon) &= \frac{\epsilon}{4} \\ P_2(\epsilon) &= \frac{\epsilon^2}{16} + \frac{\epsilon}{4}(2l+3-2ia) + ia(l+\frac{3}{2})(2l+3-M) \end{aligned} \tag{2.17}$$

For the case, $M = 2l+3$, $P_1(\epsilon)$ is the critical Polynomial[11]. On demanding $P_1(\epsilon) = 0$ correctly yields the QES energy eigenvalue as given by Eq. (2.8). On the other hand, for $M = 2l+5$, $P_2(\epsilon)$ is the critical Polynomial, and demanding $P_2(\epsilon) = 0$ correctly yields the QES energy as given by Eq. (2.9).

Following Bender and Dunne [11], it is easy to compute the norm (γ_n) of the n-th polynomial. We find

$$\gamma_n = (4ai)^n n! \prod_{k=1}^{n} (k+l+1/2)(M-2k-2l-1) \tag{2.18}$$

Weight factors (ω_1 and ω_2)[11], for $M = 2l+5$ are

$$\begin{aligned} \omega_1 &= \frac{2l+3-2ai}{\sqrt{(2l+3)^2-4a^2}} + \frac{1}{2} \\ \omega_2 &= -\frac{2l+3-2ai}{\sqrt{(2l+3)^2-4a^2}} + \frac{1}{2} \end{aligned} \tag{2.19}$$

Moments of weight function is defined by

$$\mu_n = \int dE\omega(E)E^n \tag{2.20}$$

It is easily shown that the n-th moment, for large n, is proportional to $(M+a^2)^n$.

One can similarly study the properties of the Bender-Dunne polynomials in the other three cases.

III. PERIODIC PT INVARIANT QES SYSTEM

As has been shown in [13], if under the anti-isospectral transformation $x \longrightarrow ix \equiv y$, the potential $v(x) \longrightarrow \bar{v}(y)$ and if the potential $v(x)$ has m QES levels with energy

eigenvalue and eigenfunctions E_k $(k = 0, 1, 2 \cdots m-1)$ and $\psi_k(x)$ respectively then the energy eigenvalues of $\bar{v}(y)$ are given by

$$\bar{E}_k = -E_{m-1-k}, \quad \bar{\psi}_k(y) = \psi_{m-1-k}(ix) \tag{3.1}$$

Under this anti-isospectral transformation,$x \longrightarrow ix \equiv \theta$, it is easily seen that the Hamiltonian (2.1) goes over to

$$H = p^2 + [a\cos(2\theta) - iM]^2 + \frac{l(l+1)}{\sin^2(\theta)} + \frac{l(l+1)}{\cos^2(\theta)}\,, \tag{3.2}$$

with $-1 < l < 0$. In this case $U(x) = U(x+\pi)$. This complex Hamiltonian is invariant under combined Parity [$x \longrightarrow \frac{\pi}{2} - x$] and Time reversal [$t \longrightarrow -t$ &$i \longrightarrow -i$]. Explicit QES solutions can be obtained from the solutions of hyperbolic case discussed in detail in section 2.1 by using the anti-isospectral transformation as given in Eq. (3.1). One can easily construct the BD polynomials and their properties for this case by following the methods outlined in section 2.2.

IV. REAL QES SYSTEMS

Before completing this paper, it may be worthwhile to point out that, starting from the Hermitian, QES, DSHG potential $V(x) = [a\cosh(2x) - M]^2$, one can add several perturbing terms and the resulting Hamiltonian are all examples of QES systems. In this section, we consider three such perturbing terms.

Case I: Perturbation term, $V_1 = \frac{l(l+1)}{\sinh^2(x)}$ with $-1 < l < 0$.

The combined perturbed system is described by the Hamiltonian,

$$H_I = p^2 + [a\cosh(2x) - M]^2 + \frac{l(l+1)}{\sinh^2(x)}\,, \tag{4.1}$$

We show that for integral values of $M - l - 1$ or $M + l$, this is a QES problem and one can obtain p QES eigenstates in case $M = l + 1 + p + s$ or $M = -l + p + s$ respectively, where $p = 1, 2, 3, ...$ and $s = 0, 1$.

We substitute

$$\psi(x) = e^{-\frac{a}{2}\cosh(2x)}\phi(x)\,, \tag{4.2}$$

in the Schroödinger equation $H\psi = E\psi$ with H as given by Eq. (4.1) and obtain

$$\phi''(x) - 2a\sinh(2x)\phi'(x) + [(E - M^2 - a^2) \\ +2(M-1)a\cosh(2x) - \frac{l(l+1)}{\sinh^2(x)}]\phi(x) = 0 \tag{4.3}$$

On further substituting

$$\phi = [\sinh(x)]^{\alpha}\eta\,, \tag{4.4}$$

we obtain

$$\eta''(x) + 2[\alpha\coth(x) - a\sinh(2x)]\eta'(x) + [E - M^2 - a^2 \\ +\alpha^2 - 2(M-1)a + 4(M-\alpha-1)a\cosh^2(x)]\eta(x) = 0 \tag{4.5}$$

provided

$$\alpha(\alpha - 1) = l(l+1)\,. \tag{4.6}$$

Eq.(4.6) implies either $\alpha = l+1$ or $\alpha = -l$. We first consider $\alpha = l+1$ and then it is easy to see that Eq. (4.5) with $\alpha = l+1$ has p QES solutions in case $M = l+1+p+s$ where $s = 0$ or $s = 1$. In particular the solutions are of the form

$$\eta = [\cosh(x)]^s \sum_{n=0}^{\infty} a_n[\cosh^2(x)]^n\,, \tag{4.7}$$

where $s = 0$ in case $M = l+2p$ while $s = 1$ in case $M = l+2p+1$. Few low lying solutions are

$$M = l+2\,,\quad \eta = constant\,,\quad E = a^2 + 2a(l+1) + 2l + 3\,, \\ M = l+3\,,\quad \eta = \cosh(x)\,,\quad E = a^2 + 2al + 2l + 5\,, \tag{4.8}$$

$$M = l+4\,,\quad E = y + a^2 + 3(2l+5) + 2(l+3)a\,, \\ \eta = A\cosh^2(x) + B\,,\quad y = -2(l+2+2a) \pm 2\sqrt{(l+2+2a)^2 - 4a}\,. \tag{4.9}$$

$$M = l+5\,,\quad E = y + a^2 + 8(l+3) + 2(l+4)a\,, \\ \eta = A\cosh^3(x) + B\cosh(x)\,, \\ y = -(4l+9+8a) \pm 2\sqrt{(l+2a)^2 + 3(2l+3)}\,. \tag{4.10}$$

The results for $\alpha = -l$ are immediately obtained from above by replacing everywhere l by $-l-1$.

By making the substitution $\cosh^2(x) = t$ in Eq. (4.5), one can show that the corresponding Bender-Dunne polynomials satisfy three term recursion relation.

Case II: Perturbation term, $V_2 = -\frac{l(l+1)}{\cosh^2(x)}$.

The Hamiltonian of the system is thus given by

$$H_2 = p^2 + [a\cosh(2x) - M]^2 - \frac{l(l+1)}{\cosh^2(x)}\,, \tag{4.11}$$

where l is any real number. We again show that for integral values of $M-l-1$ or $M+l$, this is a QES problem and one can obtain p QES eigenstates in case either $M = l+1+p+s$ or if $M = -l+p+s$ respectively, where $p = 1, 2, 3, ...$ and $s = 0, 1$.

We substitute

$$\psi(x) = e^{-\frac{a}{2}\cosh(2x)}\cosh^\alpha(x)\eta, \tag{4.12}$$

in the Schroödinger equation $H_2\psi = E\psi$ with H_2 as given by Eq. (2.14) to obtain

$$\begin{aligned} &\eta''(x) + 2[\alpha\tanh(x) - a\sinh(2x)]\eta'(x) + [E - M^2 - a^2 \\ &+2a(2\alpha - M + 1) + \alpha^2 + 4(M - \alpha - 2)a\cosh^2(x)]\eta(x) = 0\,, \end{aligned} \tag{4.13}$$

provided

$$\alpha(\alpha - 1) = l(l+1)\,. \tag{4.14}$$

This implies $\alpha = l+1$ or $\alpha = -l$. let us consider first, $\alpha = l+1$. It is easy to see that Eq. (4.13) with $\alpha = l+1$ has p qes solutions in case $M = l+1+p+s$ where $s = 0$ or $s = 1$. In particular the solutions are of the form

$$\eta = [\sinh(x)]^s \sum_{n=0}^{\infty} a_n[\sinh^2(x)]^n\,, \tag{4.15}$$

where $s = 0$ in case $M = l+2p$ while $s = 1$ in case $M = l+2p+1$. Few low lying solutions are

$$\begin{aligned} &M = l+2\,, \quad \eta = constant\,, \quad E = a^2 - 2a(l+1) + 2l + 3\,, \\ &M = l+3\,, \quad eta = \sinh(x)\,, \quad E = a^2 - 2al + 2l + 5\,, \end{aligned} \tag{4.16}$$

$$\begin{aligned} &M = l+4\,, \quad E = y + a^2 + 3(2l+5) - 2a(l-1)\,, \\ &\eta = A\cosh^2(x) + B\,, \\ &y = -2(l+2+2a) \pm 2\sqrt{(l+2+2a)^2 - 4a(2l+3)}\,. \end{aligned} \tag{4.17}$$

$$
\begin{aligned}
&M = l + 5\,, \quad E = y + a^2 + 8(l+3) - 2a(l-2)\,, \\
&\eta = A\sinh^3(x) + B\sinh(x)\,, \\
&y = -(4l + 9 + 4a) \pm 2\sqrt{(l-2a)^2 + 3(2l+3)}\,.
\end{aligned}
\tag{4.18}
$$

It is easy to convince oneself that all the above solutions are still solutions with replacement of l by $-l-1$ everywhere.

The three term recursion relations for the associate polynomials is obtained in this case by making the substitution $\sinh^2(x) = t$ in Eq. (4.13).

Case III: Perturbation term, $V_3 = \frac{l(l+1)}{\sinh^2(x)} - \frac{g(g+1)}{\cosh^2(x)}$.

We now show that it is still a QES problem even if the perturbation is the sum of the two perturbations considered in Case I and Case II, i.e. consider the Hamiltonian

$$
H_3 = p^2 + [a\cosh(2x) - M]^2 + \frac{l(l+1)}{\sinh^2(x)} - \frac{g(g+1)}{\cosh^2(x)}\,, \tag{4.19}
$$

where $-1 < l < 0$ so that the singularity at $x = 0$ is not strong enough. We show that for positive integral values of either $(M-l-g-1)/2$ or $(M+l+g+1)/2$ or $(M+l-g)/2$ or $(M-l+g)/2$, this is a QES problem and one can obtain p QES eigenstates in case either $M = l+g+2p+1$ or $M = -l-g+2p-1$ or $M = -l+g+2p$ or $M = l-g+2p$, where $p = 1, 2, 3, ...$

We substitute

$$
\psi(x) = e^{-\frac{a}{2}\cosh(2x)}[\cosh(x)]^{\alpha}[\sinh(x)]^{\beta}\eta\,, \tag{4.20}
$$

in the Schroödinger equation $H_3\psi = E\psi$ with H_3 as given by Eq. (4.19) we find that QES solutions exist only when $\alpha = g+1, -g$ and $\beta = l+1, -l$. On choosing $\alpha = g+1, \beta = l+1$, we obtain

$$
\begin{aligned}
&\eta''(x) + 2[(l+1)\coth(x) + (g+1)\tanh(x) - a\sinh(2x)]\eta'(x) \\
&+ \left[y + z\cosh^2(x)\right]\eta(x) = 0\,,
\end{aligned}
\tag{4.21}
$$

where

$$
y = E - M^2 - a^2 + (l+g+2)^2 + 2a(2g - M + 3)\,, \quad z = 4a(M-l-g-3)\,. \tag{4.22}
$$

It is easy to see that Eq. (4.21) has p QES solutions in case $M = l+g+2p+1$ with $p = 1, 2, 3, ...$ In particular the solutions are of the form

$$
\eta = \sum_{n=0}^{\infty} P_n[\cosh^2(x)]^n\,, \tag{4.23}
$$

in case $M = l + g + 2p + 1$. Two of the low lying solutions are

$$M = l + g + 3\,, \quad \eta = constant\,, \quad E = a^2 - 2a(2g+3) + 2l + 2g + 5\,, \tag{4.24}$$

$$\begin{aligned} &M = l + g + 5\,, \quad \eta = A\cosh^2(x) + B\,, \\ &y = -2(l+g+3+2a) \pm 2\sqrt{(l+g+3+2a)^2 - 4a(2g+3)}\,. \end{aligned} \tag{4.25}$$

The results for the remaining three cases are immediately obtained from here by replacing (l, g) with $(-l-1, -g-1)$, or with $(-l-1, g)$ or with $(l, -g-1)$.

V. CONCLUSION

In this paper, we have shown that the known QES DSHG (and hence DSG) systems (both Hermitian and complex PT-invariant one) can be further enlarged by adding perturbations and still it continues to be a QES system. It will be interesting to look at other QES examples and obtain new QES systems by adding suitable perturbating terms.

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